Equal Sharing and the Roots of Fraction Equivalence

Susan B. Empson

How would you help a child understand that 8/24 = 1/3? What kinds of ideas and activities support that understanding? A common approach to fraction equivalence is to show that two fractional parts are the same size through matching circular or rectangular regions. This approach is limited because it isolates the idea of equivalence from other potentially enriching ideas, such as multiplication and division. From my work with fourth- and fifth-grade teachers and students, I have found that in reasoning about equivalence, children readily invent a variety of strategies that incorporate their emerging understanding of multiplication and division. These strategies contribute to a richer understanding of fraction equivalence because they interconnect different mathematical ideas (Carpenter and Lehrer 1999).

This article presents examples of children's invented equal-sharing strategies (Streefland 1991) that lay a foundation for reasoning about equivalence by connecting ideas of multiplication, division, and fractions. The examples highlight some of the different ways that children think about division and show how that thinking helps them reason about fraction equivalence. The article concludes by discussing the teacher's role in supporting, refining, and extending these interconnections and considering some of the mathematical issues involved in gener-

Susan Empson, empson@mail.utexas.edu, teaches in the mathematics and science education department of the University of Texas at Austin, Austin, TX 78712-1294. She is interested in what teachers learn from student-generated strategies.

alizing invented strategies into more powerful, symbolically driven procedures.

Using Multiplication and Division to Reason about Equivalence

Fraction instruction often begins with equal-sharing problems that are similar to the following:

> Four children are sharing 3 candy bars. If the children share the candy bars equally, how much can each child have?

Varying the numbers in these problems produces a variety of fractions as solutions, including unit fractions, such as 1/3 for three children sharing one candy bar; nonunit fractions, such as 3/4 in the candy-bar problem; improper fractions, such as 10/4 for four children sharing ten candy bars; and mixed numbers, such as 2 1/6 for six children sharing thirteen candy bars. Certain number combinations create opportunities for early discussions of fraction equivalence by comparing children's different strategies for the same situation (Empson 1995, 1999).

The examples in the following paragraphs are taken from interviews with students who participated in a three-week unit on equal-sharing problems. During instruction, the children began to use what they knew about multiplication and division to create, compare, and transform equivalent fractions. The interviews captured the children's level of competence at the end of instruction.

FIGURE 1

Nicholas's strategy for sharing eight pancakes among twenty-four children

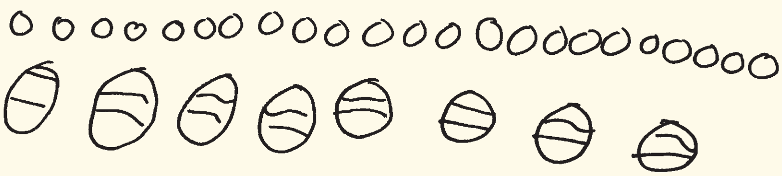
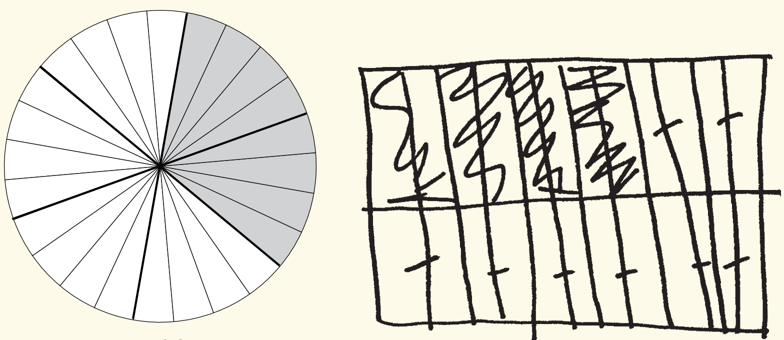


FIGURE 2

Two strategies for "chunking" smaller fractional pieces into larger ones



(a)
A schema of Erica's mental strategy

(b)
Kenza's strategy

Initial strategies: creating bigger pieces through multiplication

In solving equal-sharing problems, children seem to naturally move toward the goal of partitioning or transforming shares into the biggest possible pieces. Although their initial strategies do not readily correspond to the symbolic procedure for renaming a fraction, they do represent a productive beginning for developing an understanding of equivalence.

The following is a typical question that might be asked in the classroom:

> Twenty-four children want to share 8 pancakes so that each one gets the same amount. How much pancake can each child have?

Children might use several strategies to partition eight pancakes among twenty-four children. A common strategy involves partitioning each pancake into twenty-four pieces, which is often done mentally for such a big number, and giving each of the eight children one piece from each pancake, for a total of 8/24 of a pancake per person. In solving such problems, many students in our classroom began to wonder how to give out the largest possible pieces to the children.

For this problem, such a strategy involves partitioning the pancakes into bigger pieces than twenty-fourths. For example, Nicholas drew twenty-four children and eight pancakes. After some thinking, he decided to partition all the pancakes into thirds so that each of the twenty-four children got a third of a pancake (see **fig. 1**). He explained that he first mentally tried some other partitions—halves, fourths, and sixths—by skip counting by twos, fours, and sixes, respectively, on the eight pancakes. His goal was to create exactly twenty-four pieces of pancake, or 1 piece for each child. Other children immediately recognized 24 as a multiple of 8 and knew that they could partition each pancake into thirds.

In class discussions of such strategies, teachers drew attention to equivalence relationships by helping children understand that different partitions result in equivalent amounts when all the sharing material is used. For example, the teachers asked children how they would convince one another that 1/3 of a pancake is the same as 8/24 of a pancake. Another productive question to pose for discussion is whether Nicholas's strategy could be

used to solve equal-sharing problems that involve other numbers.

Creating bigger pieces through "chunking" smaller pieces

Nicholas made bigger pieces by thinking about how the total number of pieces could be split among the sharers; other children approached the task of making bigger pieces in other ways. For example, Erica figured that each of the twenty-four children sharing eight pancakes would get 8/24 of a pancake. Knowing that 4 was a factor of both 8 and 24, she reasoned that she could "chunk" 4 twenty-fourths together to make bigger fractional pieces (see **fig. 2a**). A share consisting of eight small pieces could be regrouped into a share consisting of two big pieces, each a sixth of a pancake in size, for a total share of 2/6 of a pancake. Other children, such as Kenza, reasoned similarly but were less proficient in multiplication. For the same problem, Kenza also figured that each sharer got 8/24, and she tried chunking every two pieces together to make bigger pieces (see **fig. 2b**). She said, "First I had to check how many would be an even number [i.e., figure out how the pieces could be divided with no remainder] to make into a different fraction. So first I tried twos and that worked, so I just stuck with the twos." This approach gave her four larger pieces per child, each 1/12 of a pancake in size. Had chunking in twos not worked, Kenza reported that she would have tried chunking every three pieces, and so on.

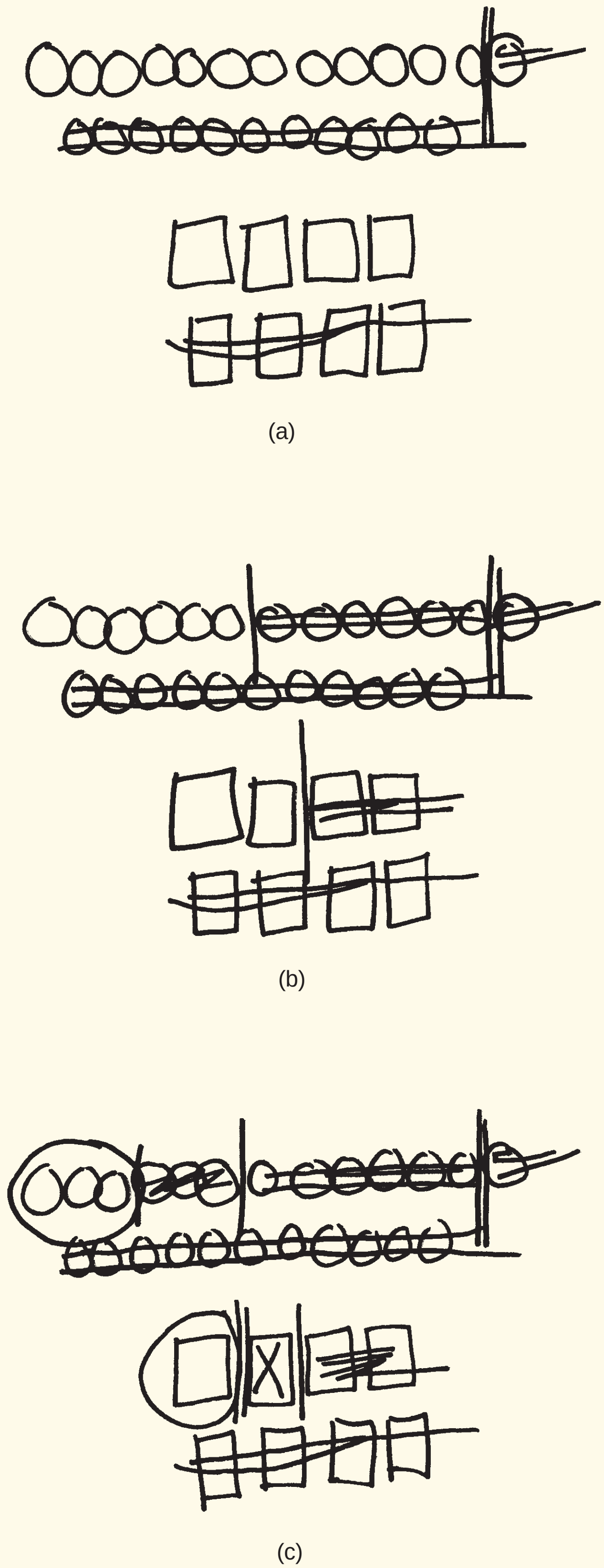
Finding equivalent fractions through partitioning the sharing situation

A completely different approach invented by children to find equivalent fractions involved partitioning both the sharing material and the children, or sharers. For example, Randy solved the pancake problem by splitting the twenty-four children and eight pancakes into two groups, crossing out one group, splitting the remaining twelve children and four pancakes into two groups, crossing out one group, and finally, splitting the remaining six children and two pancakes into two groups (see **fig. 3**). Using repeated halving, he reduced the sharing situation to three children sharing one pancake, or a third of a pancake per sharer. The implicit equivalence in his reasoning is that twenty-four children sharing eight pancakes is equivalent to three children sharing one pancake.

This strategy depends on well-chosen numbers for the initial problem, specifically, numbers that are powers of 2, such as 2, 4, 8, 16, and so on, or numbers that have factors that are powers of 2. Although even very young children intuitively use

FIGURE 3

Randy's repeated halving strategy for reducing the sharing situation

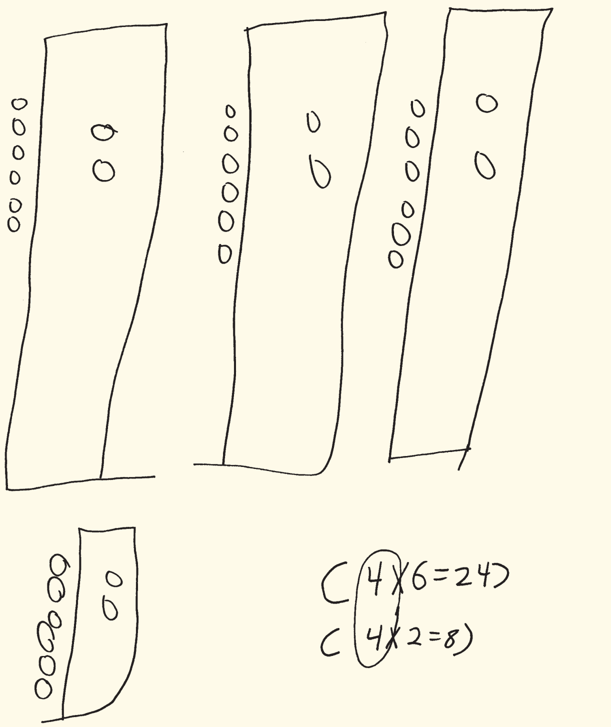


repeated halving, the approach of generalizing beyond this operation to splitting quantities into three or more groups is much less intuitive. The basis for generalizing this strategy beyond repeated halving rests in the notion of splitting sharers *and* shared objects into the same number of groups. Even children who recognize a common factor in the number of sharers and shared objects may have some trouble interpreting that factor in terms of splitting, however, because children have a tendency to interpret division in terms of a chunk of a given size (Confrey 1995). That is, children try to measure out same-sized groups rather than share among a given number of groups.

Kirk successfully used a more generalized version of repeatedly halving the sharing situation. He immediately recognized a common factor of 4 in 24 and 8 and drew four tables at which to arrange his sharers and pancakes (see **fig. 4**). He put two pancakes at each table, along with six children, and figured that each child would get two-sixths of a pancake. By partitioning both the sharers and the pancakes into the same number of groups, he reduced the situation from twenty-four children sharing eight pancakes to six children sharing two pancakes. Thinking in terms of arranging the sharers and pancakes at tables facilitated a splitting interpretation of division. Had Kirk chosen the greatest common factor, 8, his final fraction would have been in lowest terms.

FIGURE 4

Kirk split twenty-four children and eight pancakes into four groups.



Facilitating More Powerful Strategies

Clearly, students can invent equivalence-reasoning strategies that have the potential to be transformed into more generalized symbolic procedures, but that transformation is not guaranteed. The teacher must guide the transformation by acknowledging and valuing the meaning and variety of student-generated strategies in equal sharing and emphasizing a common focus on constructing strategies that work for as many different number combinations and problem contexts as possible (see, e.g., Brinker [1998]; Mack [1998]; Huinker [1998]; Lappan and Bouck [1998]).

For any invented strategy, a teacher can raise at least two mathematical questions for class discussion (Campbell, Rowan, and Suarez 1998). One is whether the strategy makes sense mathematically. The other is whether the strategy can be generalized and, if so, to what problems. Some strategies may work for all kinds of number combinations, and others, such as Randy's repeated halving or Nicholas's multiplication, may work only for particular number combinations. Figuring out what strategies can be generalized and why is an important aspect of classroom activity that supports the development of students' identities as mathematically capable people. At the same time, however, we must not underestimate the value to children of using strategies that work only for particular number combinations. Randy and Nicholas were using number relationships that they understood well in service of a mathematically significant activity, inventing strategies.

Another essential role for the teacher is to provide appropriate symbolization to represent more powerful ways of expressing contextual meanings. The important lesson to draw from equal-sharing problems is not to have students construct an exhaustive list of possible meanings for procedures with equivalent fractions but to ensure that the meanings make sense to the children and originate in the children's thinking about concepts that they understand (Hiebert et al. 1997). The progression from dividing eight pancakes among twenty-four children to understanding that 8/24 = 1/3 takes time but is aided when instruction allows concepts of equivalence, multiplication, and division to develop hand in hand. Students must have adequate time and support to construct a meaningful foundation for the thinking strategies that they will use fluently in the future. Teachers' supporting interconnections among mathematical concepts is integral to this process.

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The author would like to thank teachers Eric Pedersen and Martha Atkinson, without whom this work would not have been possible; Leander Independent School District for inviting her to work in its schools; the University of Texas at Austin for funding a summer research assignment to analyze the data on student thinking reported in this article; and the editors of Teaching Children Mathematics *for their suggestions in completing this article.* ▲